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THE STRESS CRITERION OF A TENSION MEMBER WITH  
GRADED FLEXURAL STIFFNESS

(Contribution to the Problem of "Clamping Effect" Outside  
of the Elastic Range)

By Hans W. Kaul

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## SUMMARY

The approximate size of the stress criterion of a bar on two supports stressed beyond the elastic range is assessed by an approximation. The calculation proceeds from the premise of "substitute flexural stiffness" so defined that the part stressed beyond the elastic range may be considered as following Hooke's law when determining the flexural deformation quantities. For the determination of the substitute flexural stiffness, it is presumed that the material is already stressed so much beyond the yield point as to be strain-hardened. The data are directly applicable to materials having no definite yield point. For the rest, von Karman's method for compressed and subsequently deflected bars serves as basis for the calculation.

The action of bars on reaching the yield point is not discussed; it is to form the subject of a special report.

As regards the magnitude of the stress criterion of tension bars outside of the elastic range within the stress region - yield point and ultimate stress, wherein the substitute flexural stiffness is quite small and finally approaches zero, the following may be stated: If the whole bar is stressed beyond the elastic range, the stress criterion itself is very small and approaches zero near the ultimate load.

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\*"Die Spannziffer eines Zugstabes mit abgestufter Biegesteifigkeit. (Ein Beitrag zur Frage der 'Einspannwirkung' im ausserelastischen Bereich.)" Luftfahrtforschung, vol. 13, no. 6, June 20, 1936, pp. 181-189.

If an elastic margin remains near the point of moment application (nodal point), such as may happen through thickening of the part when some other welds adjoin, the stress criterion is considerably higher and remains of finite magnitude even near the ultimate load. The amount of increase depends upon the length of the elastic margin.

Neither the total length of the bar nor the length of the part stressed beyond the elastic range has any appreciable effect (for example, the heated zones in welds), so far as it does not pertain to very small lengths which are of minor practical importance.

The effect of the kind of support at the opposite end of the bar from the joint itself is considered only when the length of the part stressed beyond the elastic range is very small. The final results are:

The elastic margin has infinitely great flexural stiffness;

The part stressed beyond the elastic range has infinitely great length.

## I. INTRODUCTION

Supplementing a previous report (reference 1), the present consideration based on theory only is extended to the approximate magnitude of the stress criterion for a bar on two supports stressed beyond the proportionality limit.

In particular, the case is explored where the bar, under constant axial load throughout its length, is stressed only over part of it beyond the elastic range.

Such a case occurs in welded structures when material concentration in the joints results in cross sections greater than in the free length of the member; and again, when in the heated zones around a welded joint, the yield limit and the proportionality limit are locally much reduced (reference 2).

The calculation of the stress criterion is reduced to an equivalent elastic member (i.e., following Hooke's law), a "substitute bending stiffness" for a section length

stressed beyond the yield limit being so defined as to permit its insertion in the solution of the differential equation for an elastic member of identical length and support conditions. For the calculation of the substitute bending stiffness it is, moreover, assumed that the member is already stressed so much beyond the yield point as to have become strain-hardened again. The conditions at the yield point itself are reserved for a future report.

As regards approximate size in particular, it is attempted to establish how the stress criterion at the end of a member with two supports changes with the length of the part stressed beyond the elastic range and with its location in the member.

## II. THE STRESS CRITERION OF AN ELASTIC TENSION MEMBER ON TWO SUPPORTS WITH GRADED BENDING STIFFNESS AND CONSTANT AXIAL LOAD

### A. General

From the approximate differential equation for the elastic line of a bar stressed in tension and bending

$$M(x) - S y = - E J \frac{d^2 y}{dx^2}$$

$y$  is deflection

$S$ , stress in member (assumed constant throughout its length)

$M(x)$ , total moment of external loads and individual moments at portion separated at  $x$ , relative to the section at  $x$  of the assumedly nondeformed member, it is possible to compute (reference 3) the stress criterion

$$m = \frac{M_i}{\psi_i}$$

$M_i$  is the individual outside bending moment at point  $i$

$\psi_i$ , torsion of section at point  $i$  due to  $M_i$ , at any point  $i$  of the member under any limiting conditions.

B. Stress Criterion of an Elastic Bar Pin-Jointed at Both Ends on Two Supports with Two and Three Sections of Varying Flexural Stiffness

1. Bars with two sections having unlike flexural stiffness. - Figure 1 shows a bar pin-jointed at both ends supported at two points, 0 and 2, and stressed under constant axial load  $S$  and with a flexural end moment  $M_0$ ; its flexural stiffness along panel 0 to 1 is constantly equal to  $EJ_1$ , and along panel 1 to 2, constantly equal to  $EJ_2$ .

$$\text{With } k_i = \sqrt{\frac{EJ_i}{S}} \quad \text{and} \quad \alpha_i = \frac{s_i}{k_i}$$

$$U_i = \frac{k_i}{s} \tanh \alpha_i \quad \text{and} \quad V_{ik} = \frac{U_k}{\cosh^2 \alpha_i (U_i + U_k)}$$

and observance of the limiting conditions at the supports 0 and 2 and of the stability conditions at point 1, the stress criterion at support 0 follows from the approximate differential equation for the elastic line in panels 0 to 2 at:

$$m^{0.2} = \frac{S s U_1}{1 - U_1 - V_{1.2}} \quad (1)$$

Table of formulas 1 gives the limiting values approached by  $m^{0.2}$  when one or more of  $s_1$ ,  $s_2$ ,  $EJ_1$ ,  $EJ_2$  become zero or infinitely great. With  $s_1 = 0$  or  $s_2 = 0$  the value of the stress criterion of a pin elastic bar on two supports with constant flexural stiffness is obtained, while  $s_1 = \infty$  affords the corresponding limit value of the bar with constant flexural stiffness for  $s = \infty$ .

Table of formulas 1. - Several limiting values of  $m^{0.2}$  corresponding to formula (1) for extreme values of the quantities  $s_1, s_2, EJ_1, EJ_2$ .

No.	$s_1$	$s_2$	$EJ_1$	$EJ_2$	$m^{0.2}$
1	0	$s$	$EJ_1$	$EJ_2$	$\frac{s S}{\frac{\alpha_2}{\tanh \alpha_2} - 1}$
2	$s$	0	$EJ_1$	$EJ_2$	$\frac{s S}{\frac{\alpha_1}{\tanh \alpha_1} - 1}$
3	$\infty$	$s_2$	$EJ_1$	$EJ_2$	$\sqrt{S E J_1}$
4	$s_1$	$\infty$	$EJ_1$	$EJ_2$	$1 - \frac{S s U_1}{\cosh^2 \alpha_1 \left( \frac{s U_1}{k_2} + 1 \right)}$
5	$s_1$	$\infty$	$EJ_1$	0	$S s U_1$
6	$s_1$	$\infty$	$\infty$	$EJ_2$	$S(s_1 + k_2)$
7	$s_1$	$s_2$	$\infty$	0	$s S \frac{s_1}{s_2}$
8	$s_1$	$\infty$	$\infty$	0	$S s_1$

2. Member with three sections of different flexural stiffness (fig. 2). - The following abbreviation is introduced:

$$W_{1.3} = \frac{U_2}{\cosh^2 \alpha_1 \left[ U_1 + U_2 - \frac{U_1 U_3}{\cosh^2 \alpha_2 (U_2 + U_3)} \right]}$$

As in the preceding paragraph, we obtain for point 0 the stress criterion:

$$m^{0.3} = \frac{S s U_1}{1 - U_1 - W_{1.3}} \quad (2)$$

Table of formulas 2 contains the limiting values approached by  $m$  when one or more of  $s_1$ ,  $s_2$ ,  $s_3$ ,  $EJ_1$ ,  $EJ_2$ ,  $EJ_3$  become zero or infinitely great. With  $s_1 = 0$ ,  $s_2 = 0$ , or  $s_3 = 0$ , we obtain the stress criterion for an identically supported and loaded bar with two sections of different flexural stiffness, and with  $s_2 = \infty$  the corresponding limiting value of section 1.

The limiting values of tabulations 1 and 2 are subsequently employed in the calculation of the stress criterion of welded steel tube trusses, where the practical importance of individual cases is discussed in detail. For the present it suffices to state that the flexural stiffness of parts of the length (heat zones) stressed beyond the elastic range is so small compared to the elastically constant parts that, particularly vicinal to breaking stress, the elastically constant sections may be ascribed the term "infinite" flexural stiffness, and those stressed beyond the elastic range, the term "zero" flexural stiffness.

### III. ROUGH CALCULATION OF STRESS CRITERION FOR THE CASE OF ONE SECTION OF THE BAR BEING STRETCHED BEYOND THE YIELD POINT

#### A. Method of Calculation

Flexural deformations of members in compression beyond the elastic range have been usually described in literature by giving "substitute flexural stiffness"  $EJ_{\text{sub}}$  a suitable definition so as to permit its direct insertion in the differential equation of an elastic bar (i.e., following Hooke's law) of the same length and the same support conditions. So the calculation of a bar stressed beyond the elastic range reduces to the case of an "equivalent elastic bar." With this procedure the zone at the yield point was generally disregarded altogether or else bridged over by interpolation. This range is theoretically and experimentally treated for bars stressed in combined tension and bending in a report to be published very soon.

The present fundamental investigation also disregards the range on the yield point  $\sigma_F$  in relation to the approximate magnitude of the stress criterion; the "substitute flexural stiffness" of the "equivalent elastic bar" is computed for the case of simultaneous occurrence of very great tension and small flexural stresses under the assumption that the flexural and moment  $M_0$  is applied only at a tensile stress  $\sigma_m = \frac{S}{F} > \sigma_F$ , at which the material of the portion stressed beyond the elastic range has already become strain-hardened. For materials on which no yield point is discernible, the following considerations are, of course, summarily applicable.

## B. The Substitute Flexural Stiffness

(v. Kármán's method (reference 4))

Following a suggestion by Engesser, v. Kármán determines the substitute flexural stiffness of a bar compressed and then deflected beyond the elastic range on the following premises.

1. Under minor flexure of a straight bar the elongations of the individual fiber correspond even beyond the elasticity limit to the same stresses produced by these elongations under pure compression.
2. On deflection flat sections shall remain flat.

On these premises, v. Kármán computes a "resultant deflection modulus" which, multiplied by the particular equatorial moment of inertia of the cross-sectional area relative to its gravity axis, gives the substitute flexural stiffness for a definite main axis plane, and to which the following relation is applicable:\*

$$C_{res} = \frac{J_1}{J} C_{total} + \frac{J_2}{J} C_{elastic} \quad (3)$$

Applying v. Kármán's definitions for bars compressed beyond the elastic range to bars stressed in tension beyond the yield point, the notations in equation (3) designate:

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\*von Kármán denotes the resultant modulus  $C_{res}$  with  $M$ .



$C_{total} = \frac{d\sigma}{d\epsilon_{total}}$  is the modulus of the total form changes, defined by the slope of the tangent placed on the stress-strain curve  $\sigma = f(\epsilon_{total})$  at point  $(\sigma_m; \epsilon_m)$ ;  $\sigma_m$  is the pure tensile strain  $S/F$  on which the flexural stresses are superposed;  $\epsilon_m$  is the elongation relative to  $\sigma_m$  in the tension test.

$C_{elastic} = \frac{d\sigma}{d\epsilon_{elastic}}$  is the modulus of the elastic deformations, defined by the slope of that tangent on the return curve of the hysteresis loop in a tensile test with a stress  $\sigma_m$  as peak of the loop which touches the return curve at the peak (point  $\sigma_m; \epsilon_m$ ).

$J$  is the moment of inertia of the bar section with respect to its axis of gravity.

$J_1$ , the moment of inertia of the section part lying on the tension side of the flexural stresses relative to the "neutral" axis, which is defined as that straight line lying in the section along which the additional stresses from the flexure are zero.

$J_2$ , the corresponding moment of inertia of the section part lying on the compression side of the flexural stresses.

The substitute flexural stiffness  $(EJ)_{sub}$  is equal to the product of resultant deformation modulus and inertia moment  $J$

$$(EJ)_{sub} = C_{res} J = J_1 C_{total} + J_2 C_{elastic} \quad (4)$$

The position of the neutral axial is defined by:

$$\underline{S}_1 C_{total} = \underline{S}_2 C_{elastic} \quad (5)$$

wherein  $\underline{S}_1$  and  $\underline{S}_2$  are the static moments corresponding to the inertia moments  $J_1$  and  $J_2$ .

Table of formulas 2. - Several limiting values of  $m^{0.3}$  corresponding to formula (2) for extreme values of the quantities  $s_1, s_2, s_3, EJ_1, EJ_2, EJ_3$ .

No.	$s_1$	$s_2$	$s_3$	$EJ_1$	$EJ_2$	$EJ_3$	$m^{0.3}$
1	0	$s_2$	$s_3$	$EJ_1$	$EJ_2$	$EJ_3$	$\frac{S s U_2}{1 - U_2 - V_{2.3}}$
2	$s_1$	0	$s_3$	$EJ_1$	$EJ_2$	$EJ_3$	$\frac{S s U_1}{1 - U_1 - V_{1.3}}$
3	$s_1$	$s_2$	0	$EJ_1$	$EJ_2$	$EJ_3$	$\frac{S s U_1}{1 - U_1 - V_{1.2}}$
4	$\infty$	$s_2$	$s_3$	$EJ_1$	$EJ_2$	$EJ_3$	$\sqrt{S EJ_1}$
5	$s_1$	$\infty$	$s_3$	$EJ_1$	$EJ_2$	$EJ_3$	$\frac{S s U_1}{1 - \frac{1}{\cosh^2 \alpha_1 \left( \frac{U_1 s}{k_2} + 1 \right)}}$
6	$s_1$	$s_2$	$\infty$	$EJ_1$	$EJ_2$	$EJ_3$	$1 - \frac{S s U_1}{s U_2 \cosh^2 \alpha_1 \left[ s U_1 + s U_2 - \frac{s U_1}{\cosh^2 \alpha_2 \left( \frac{U_2 s}{k_3} + 1 \right)} \right]}$
7	$s_1$	$s_2$	$\infty$	$EJ_1$	0	$EJ_3$	$\frac{S s U_1}{s U_2}$
8	$s_1$	$s_2$	$\infty$	$\infty$	$EJ_2$	$EJ_3$	$1 - \frac{S s_1}{s U_2 + \frac{s_1 k_3}{\cosh^2 \alpha_2 (s U_2 + k_3)}}$
9	$s_1$	$s_2$	$\infty$	$\infty$	$EJ_2$	$\infty$	$1 - \frac{S s_1}{s U_2 + \frac{s_1}{\cosh^2 \alpha_2}}$
10	$s_1$	$s_2$	$\infty$	$\infty$	0	$\infty$	$\frac{S s_1}{S s s_1}$
11	$s_1$	$s_2$	$s_3$	$\infty$	0	$\infty$	$\frac{S s s_1}{S - s_1}$

A precise determination of the form changes, especially following bending moments of finite magnitude, requires several generalizations relative to v. Kármán's calculation method; in particular, it should, as shown by W. Rein (reference 5), include the previous history of the explored piece. But as these refinements afford only minor improvements while making the calculation more protracted, they are disregarded in the present report.

### C. Approximate Magnitude of the Substitute Flexural

#### Stiffness for a Certain Material

In order to get an idea of the approximate size of the substitute flexural stiffness still to be expected from bars stressed in tension beyond the yield point, the Appendix of this paper contains the substitute flexural stiffness of bars fabricated from a certain carbon steel and in particular, of circular sections frequently used in airplane design as welded structures. The Appendix also contains detailed data regarding the particular material and its stress-strain curve in prestretched condition during drawing and in the annealed state during welding. It further is shown that in the annealed zones on welds of the chosen steel, the breaking strain is reached at tensile stresses at which the blank-drawn material (i.e., that considerably stretched during drawing) still reveals no appreciable discrepancy from Hooke's law. In the investigation of the deformations of welds from the particular carbon steel, which included both blank-drawn and annealed pieces, the blank-drawn parts of the bars may still be assumed elastic when the annealed parts are already under tension beyond the elastic range.

The deformation moduli established from tension tests have the following values:

1. Blank-drawn tubing -  
(Mean value for elastic range approximately up to ultimate stress of annealed material):

$$C_{\text{elastic}} = C_{\text{total}} = E = 1.85 \times 10^6 \text{ kg/cm}^2$$

2. Annealed tubing -  
Elastic range (mean value, valid approximately up to yield point):

$$C_{\text{elastic}} = C_{\text{total}} = E = 1.97 \times 10^6 \text{ kg/cm}^2$$

TABLE III. Outside of Elastic Range  
(after strain-hardening at  $\sigma \approx 3,000$  kg/cm<sup>2</sup>)

$\sigma$ kg/cm <sup>2</sup>	$10^{-6} C_{\text{total}}$ kg/cm <sup>2</sup>	$10^{-6} C_{\text{elastic}}$ kg/cm <sup>2</sup>
3,000	0.0180	1.65
3,350	.0114	1.65
3,670	.0027	1.40

This affords for steel tubes of 20/1 and 30/0.5 diameter/wall thickness ratio the comparative  $EJ_{\text{sub}}/EJ$  values in the elastic range of table IV. The last column gives the  $EJ_{\text{sub}}/EJ$  values for full rectangular sections in comparison. The effect of the cross-section form is seen to be relatively small. (v. Kármán and others obtain the same result by comparing full rectangular and I section.)

TABLE IV

Mean tensile stress  m	$\frac{EJ_{\text{sub}}}{EJ}$ for following cross-sectional forms			Deformation moduli (kg/cm <sup>2</sup> )	
	Ring		Complete rectangle	$10^{-6}$	$10^{-6}$
	D/ $\delta$ =20/1	D/ $\delta$ =30/0.5		$C_{\text{elastic}}$	$C_{\text{total}}$
3,000	0.0289	0.0283	0.0313	} ~ 1.65	0.0180
3,350	.0183	.0178	.0204		.0114
3,670	.0043	.0043	.0053		.0027

D. The Stress Criterion of a Straight Round Tube on  
Two Supports with Varying Length and Location  
of the Part Stressed beyond the Yield Point

1. Method of calculation. - The part stressed beyond the elastic range is given a substitute flexural stiffness  $EJ_{\text{sub}}$ .

Proceeding from the bar with two segments of unlike flexural stiffness, it is attempted to ascertain the change in stress criterion for different length conditions of the two segments - once when the segment lying closest to the joint is stressed within, and then when stressed beyond the elastic range.

The premise is that the bending moment  $M$  is applied only at the pertinent tensile stress  $\sigma_m$ , so that the previous history of the piece need not be considered.

2. Bar pin-jointed at both ends on two supports with two segments of unlike flexural stiffness (fig. 3). - The test specimen was a round tube  $D/\delta = 20/1$ , 17 cm long. The results obtained herewith are equally of general importance for other lengths.

Figure 4 gives the stress criterion  $m^{0.2}$  according to formula (1) for various lengths  $s_1$  versus the mean tensile stress  $\sigma_m = S/F$ . Even a small elastic margin  $s_1$  at the joint results, as is seen, in a substantial rise in stress criterion. (Compare the curve for  $s_1 = 1$  cm with that for  $s_1 = 0$ .) On approaching the ultimate load, that is, in the case  $\sigma_m \approx \sigma_{\text{failure}}$  i.e.,  $EJ_2 = EJ_{\text{sub}} = 0$ ,  $s_1 = 0$  also gives  $m^{0.2} = 0$ , whereas finite values  $s_1$  themselves are supplemented by finite values of the stress criterion  $m^{0.2}$ , which increase in greater proportion as length  $s_1$  increases. This is shown in figure 5, where  $m^{0.2}$  is plotted against  $s_1$  for various  $EJ_2$ .

This fact is essential for the type of dependence of clamping factor on the stress criterion, for it was proved in another report (reference 1) that the clamping factor  $\bar{S}_{\text{crit}} : \bar{S}_E$ , that is, the ratio of buckling load  $\bar{S}_{\text{crit}}$  of a two-bar group consisting of tension and compression member to "natural" buckling load  $\bar{S}_E = \frac{\pi^2 EJ}{S}$  of the compression street, increases at first very steeply with increasing stress criterion  $m$  of the tension member from 1.0 for  $m = 0$  and then asymptotically approaches the limiting values 2.045 for  $m = \infty$  (rigid restraint). The range of small  $m$  with its steep rise in clamping factor being important from the practical standpoint, a small elastic margin  $s_1$  in a bar stressed in tension beyond the elastic range can raise the clamping factor, i.e., the buckling load of a group of bars near the point of restraint, quite materially.

An example of such a margin  $s_1$  on a welded joint is given in figure 6. The presence of welded angles or stirrups in two mutually perpendicular planes may afford elastic margins of greater length  $s_1$ .

The segment  $s_1$  closest to the joint is stressed beyond the elastic range, while the other segment  $s_2$  is not (fig. 7).

Figure 8 gives the stress criterion  $m^{0.2}$  versus the mean tensile stress  $\sigma_m = S/F$  for various lengths  $s_1$  of the segment stressed beyond the elastic range. On increasing the length of  $s_1$  the curves quickly approach the lower limiting curve for  $s_1 = s$ ,  $s_2 = 0$  (the whole bar stressed beyond elastic range). In our case the curves for  $s_1 = 3 \text{ cm} = 1.5 D$  and for  $s_1 = s$  already differ only little, i.e., the hot zones developed in gas fusion welding are themselves often sufficient to cause the stress criterion to drop almost to the lowest possible limiting value (fig. 11).

3. Bars having three segments of different flexural stiffness. - The examples here apply to cases where the segment of length  $s_1$  closest to the joint is stressed elastically, the next one  $s_2$  stressed beyond the elastic range (heated zone), and the one next to the other support ( $s_3$ ), is again stressed elastically (fig. 9).

In figure 10 the stress criterion  $m^{0.3}$  is plotted against the mean tensile stress  $\sigma_m = S/F$  for  $s_1 = \text{constant} = 1 \text{ cm}$ ,  $s = \text{constant} = 17 \text{ cm}$ ,  $D/\delta = 20/1$  and different values of  $s_2$ . The case is like that treated last in the preceding section; even figure 10 is fundamentally like figure 8 - that is, by existence of an elastic margin of given length  $s_1$  at the joint, a length increase of the next following segment stressed beyond elastic range by a certain amount (here  $s_2 \geq 1.5 D$ ) has practically no effect on the stress criterion. This is particularly evident in figure 11, where the stress criterion  $m^{0.3}$  is shown against length  $s_2$  for different values  $\sigma_m$ .

Plotting the stress criterion for constant values  $s_2 = \text{constant}$  and  $s = \text{constant}$  against  $\sigma_m$  with  $s_1$  as parameter, results in a set of curves which fundamentally is similar to that in figure 4; that is, even in a bar conformable to figure 9, an elastic margin affects the stress criterion at the joint approximately as its length.

4. Effect of type of support at the end of the bar opposite to the joint of applied moment. - This effect is given only a cursory treatment in the preceding example of

a bar with three sections of constant flexural stiffness. In the extreme case  $EJ_1 = EJ_3 = \infty$ ,  $EJ_2 = 0$  which, as regards approximate size, corresponds to the case where sections  $s_1$  and  $s_3$  are stressed elastically, and section  $s_2$  is stressed beyond the elastic range, we have in the case of hinged end support on both sides:

$$m = \frac{S}{s_2} \frac{s_1}{s_2 + s_3} \quad (\text{table of formulas 2})$$

If section  $s_3$  is rigidly restrained in end point 3, it is in the present case equivalent to a shortening of the bar by an amount  $s_3$  (table of formulas 2, fig. 11, and table of formulas 1, fig. 7), so that:

$$m = \frac{S (s_1 + s_2) s_1}{s_2^2}$$

Table V gives for a load  $S = 2,000$  kg, an elastic margin at the point of moment application  $s_1 = 1$  cm and different bar lengths  $s$  the values  $m^{0.3}$  for hinged and clamped support of bar end 3. In both cases the values approach each other quickly as  $s_2$  increases.

TABLE V.  $S = 2000$  kg,  $s_1 = 1$  cm,  $EJ_1 = EJ_3 = \infty$ ,  $EJ_2 = 0$

$s_2$ m	Rigid restraint at end 3 $s = 17$ cm and $s = 47$ cm or $s = \infty$	Hinged support at end point 3		
		$s = 17$ cm	$s = 47$ cm	$s = \infty$
1	4,000	2,125	2,040	2,000
2	3,000			
3	2,660			
4	2,500			
5	2,400			
6	2,340			
7	2,250			

This table further reveals the effect of total bar length  $s$  on the size of  $m$  to be small and even zero in special, extreme cases; consequently, the data obtained previously with a bar of 17 cm total length is of general validity for any length.

5. Simple approximate calculation of the stress criterion of bars stressed in tension beyond the elastic range. - This affords the possibility for a very simple approximate calculation of the stress criterion of bars stressed beyond the elastic range, which have an elastic margin at the point of applied moment (such as the joint on a framework). The  $m$  values are generally sufficiently exact when the calculation bases on a bar hinged at both ends and having two sections of different flexural stiffness - the elastic part having length  $s_1$  and flexural stiffness  $EJ_1 = \infty$ ; the other, length  $s_2 = \infty$  and flexural stiffness  $EJ_2 = (EJ)_{\text{sub}}$ . Then the stress criterion is (table of formulas 1, No. 6)

$$m^{0.2} = S (s_1 + k_2)$$

Figures 12a to 12d give the stress criterion  $m^{0.2}$  of the elastic margin versus the substitute flexural stiffness of the bar section stressed beyond the elastic range for divers lengths  $s_1$ . The tension  $S$  forms the parameter. Thus it is possible to obtain the stress criterion direct or else by interpolation for a given group of values  $s_1$ ,  $S$  and  $EJ_2 = EJ_{\text{sub}}$ .

#### IV. APPENDIX

##### CALCULATION OF THE SUBSTITUTE FLEXURAL STIFFNESS OF ROUND TUBES WITH NUMERICAL EXAMPLES FOR TUBES OF CERTAIN CARBON STEEL

##### A. Calculation of the Substitute Flexural Stiffness for the Circular Section

Here it is necessary to distinguish between the two cases that the neutral axis intersects the median line of the ring perpendicular to it:

- 1) between the center of the ring and the inner wall of the tube, that is  $(R - a) > \delta$  (see fig. 13),  
or
- 2) within the tube walls, so that  $(R - a) \leq \delta$ . (See fig. 14.)



In the first case  $(EJ)_{\text{sub}}$  is readily approximated (see reference 2), while the second case requires an exact calculation.

To 1)  $(R - a) > \delta$  (fig. 13)

In this case the assumption of a part of the cross-sectional area equal to the product of the length of the relative arc of the wall center line and the wall thickness affords a close approximation. Resorting to Rechtlich's notation  $r_m = \frac{R + r}{2}$  and  $n = \frac{a}{r_m}$ , the calculation of the comparative factors  $\frac{S_1}{S_2}$  and  $\frac{J_1}{J_2}$  gives:

$$\frac{S_1}{S_2} = \frac{\sqrt{1 - n^2} + n (\pi - \arccos n)}{\sqrt{1 - n^2} - n \arccos n}$$

$$\frac{J_1}{J_2} = \frac{(\pi - \arccos n) (1 + 2n^2) + 3n \sqrt{1 - n^2}}{\arccos n (1 - 2n^2) - 3n \sqrt{1 - n^2}}$$

These two ratios are thus shown to become independent of the wall thickness  $\delta$  and to be solely dependent upon  $n$ . Now the substitute flexural stiffness  $EJ_{\text{sub}}$  for a certain mean tensile stress  $\sigma_m$  can be determined as follows: form the ratio  $\frac{C_{\text{elastic}}}{C_{\text{total}}} = \frac{S_1}{S_2}$  for the particular  $\sigma_m$ ; read from Rechtlich's curve  $\frac{S_1}{S_2} = f(n)$  the corresponding value  $n = \frac{a}{r_m}$ , which applied to the curve  $\frac{J_1}{J_2} = F(n)$  then gives  $\frac{J_1}{J_2}$ . Now all needed data for a given tube ( $r_m, F, i, J$ ) are known and  $(EJ)_{\text{sub}}$  can be obtained because equation (3) may be transformed by means of the relation:

$$J_1 + J_2 = J + F a^2$$

to

$$C_{\text{res}} = \frac{\frac{J_1}{J_2} \left(1 + \frac{a^2}{i^2}\right)}{\left(\frac{J_1}{J_2} + 1\right)} C_{\text{total}} + \frac{\left(1 + \frac{a^2}{i^2}\right)}{\left(\frac{J_1}{J_2} + 1\right)} C_{\text{elastic}}$$

and which gives

$$(EJ)_{\text{sub}} = C_{\text{res}} J$$

To 2)  $(R - a) \leq \delta$  (fig. 14).

First determine from

$$\underline{S}_1 C_{\text{total}} = \underline{S}_2 C_{\text{elastic}}$$

the position of the neutral axis by defining  $a$  and  $\varphi_0$  (fig. 14).

For  $\varphi_0$  the equation which is solvable with the aid of figure 15, reads as follows:

$$\sin^2 \frac{\varphi_0}{2} \tan \frac{\varphi_0}{2} = \frac{3\pi}{2} \frac{\left[1 - \left(\frac{r}{R}\right)^2\right]}{\left(\frac{C_{\text{elastic}}}{C_{\text{total}}} - 1\right)}$$

which then gives

$$c = R \left(1 - \cos \frac{\varphi_0}{2}\right) \text{ and } a = R - c$$

Now when computing  $(EJ)_{\text{sub}}$ , it is expedient to introduce an "ideal cross section" so that the calculation can be made with a straight stress distribution (constant deformation modulus  $C_{\text{elastic}}$ ). The width  $2f$  of each fiber (fig. 14) is shortened in the ratio  $\frac{C_{\text{total}}}{C_{\text{elastic}}}$ , leaving thus a cross section with a "substitute inertia moment"  $J_{\text{sub}}$  relative to the neutral axis for which:

$$\begin{aligned} J_{\text{sub}} = & \left(1 - \frac{C_{\text{total}}}{C_{\text{elastic}}}\right) \left[ \frac{R^4}{8} \left(\varphi_0 - \frac{1}{2} \sin 2\varphi_0\right) \right. \\ & \left. - \frac{a^2 R^2}{2} \left(\varphi_0 - \sin \varphi_0\right) \right] \\ & + \frac{C_{\text{total}}}{C_{\text{elastic}}} \left[ \frac{\pi}{4} (R^4 - r^4) + \pi a^2 (R^2 - r^2) \right] \end{aligned}$$

and finally

$$(EJ)_{\text{sub}} = C_{\text{elastic}} J_{\text{sub}}$$

## B. Numerical Examples of the Substitute Flexural Stiffness for a Certain Steel

Preparatory to the tension-bending tests beyond the elastic range, the deformation moduli  $C_{total}$  and  $C_{elastic}$  of the test material had been established by the Materials Branch of the DVL.

The material consisted of commercial blank-drawn Bohler steel tubing, having a breaking strength of from 40 to 50 kg/mm<sup>2</sup> in the blank-drawn state. The stress-strain curves had been determined on both plain and annealed specimens. The results of the annealed samples reveal the behavior of the metal in the heated zones of welds.

Figure 16 illustrates the total elongation  $\epsilon_{total}$  versus the tension  $\sigma$  for two blank-drawn sample tubes. The yield point and breaking limit of the annealed samples, which are shown also, disclose the blank-drawn tube to still approximately follow Hooke's law at a load corresponding to the breaking stress in the annealed tubes. Thus the investigation of the deformations on welds of the particular metal including both plain and annealed specimens may be made on the premise that the blank-drawn (plain) parts of the bars are stressed within the elastic range. The deformation moduli quoted in a preceding section were obtained from the test data. As will be seen in table III, the modulus of the total form change  $C_{total}$  in the range beyond the elastic limit is less than 1/100 of modulus  $E$  within the elastic limit after strain-hardening. The modulus of the elastic form change  $C_{elastic}$  is, above the yield point, still a little less than modulus  $E$  (reference 1), although it remains of the approximate size of  $E$ . The values for  $C_{elastic}$  correspond to the slope of the tangents at the peak of the hysteresis loop (compare the --- tangents in fig. 17); that is, they are maximum values with very little stress decrease. At great values of stress decrease  $C_{elastic}$  decreases along the return curve of the hysteresis loop as much as ~ 25 percent (for stress decrease to zero).

Numerical values of the substitute flexural stiffness  $(EJ)_{sub}$ . - The calculation of  $EJ_{sub}$  for circular sections is made as stated in the previous section. Table

IV gives the values  $\frac{EJ_{sub}}{EJ}$  for  $D/\delta = 20/1$  and  $D/\delta = 30/0.5$  for three different mean tensile stresses  $\sigma_m$ . Incident to the examples for round tubes in table IV, it is to be noted that the distance  $R - a$  of the neutral fiber from the compressive edge fiber (fig. 14) will be less than the wall thickness for the particular form-change moduli.

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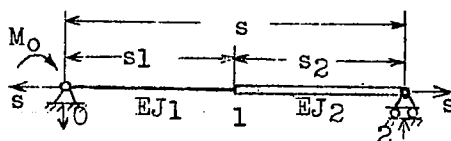


Figure 1.- Bar on two hinged supports with two parts having unlike flexural stiffness.

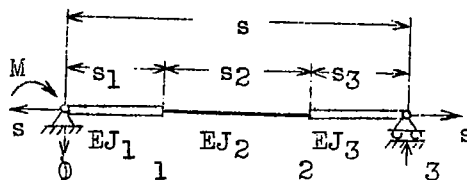


Figure 2.- Bar on two hinged supports with three parts having different flexural stiffness.

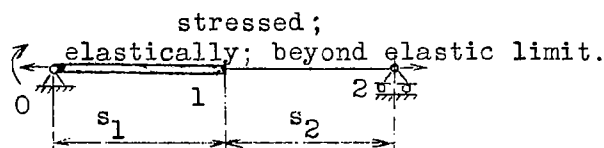


Figure 3.- Bar on two hinged supports with two parts having different flexural stiffness, one part being stressed beyond the elastic range.

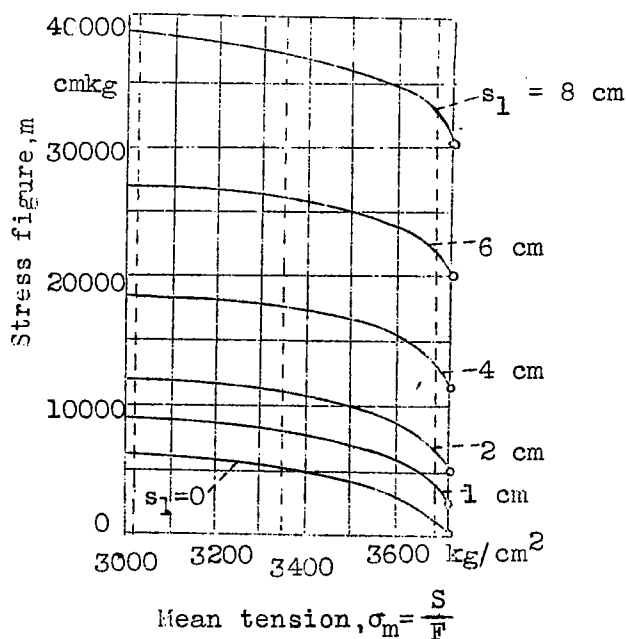


Figure 4.- Stress criterion  $m$  versus mean tension  $\sigma_m$  for different lengths  $s$ , of elastic margin at the joint (for ultimate stress  $\sigma_{Br} = 3720 \text{ kg/cm}$  it is  $EJ_2 = 0$ ).

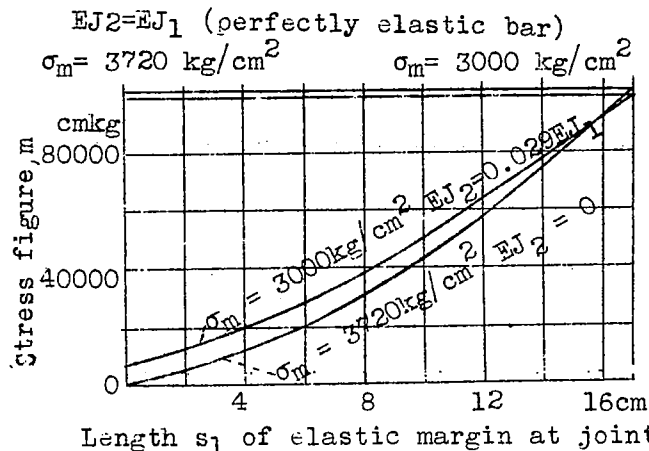


Figure 5.- m versus length  $s_1$  of elastic margin for different substitute flexural stiffness  $EJ_2$

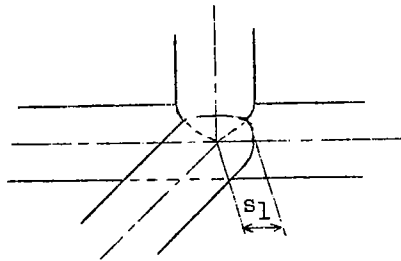


Figure 6.- Formation of an elastic margin  $s_1$  in welded joints.

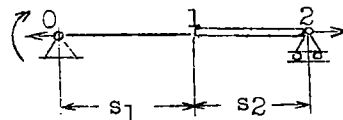


Figure 7.- Bar on two hinged supports with two sections of different flexural stiffness, the part near the joint being stressed beyond the elastic range.

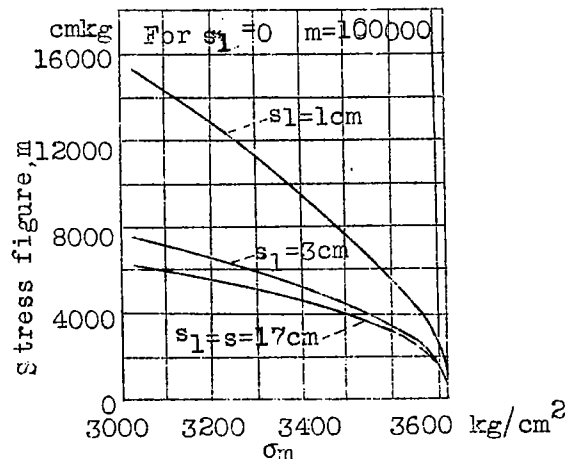


Figure 8.- Stress criterion  $m$  versus  $\sigma_m$  for different length  $s_1$  of the part stressed beyond the elastic range at the joint.

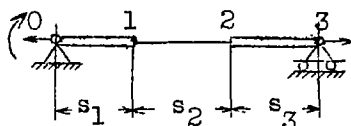


Figure 9.- Bar on two hinged supports with three sections of different flexural stiffness, the center section being stressed beyond the elastic range.

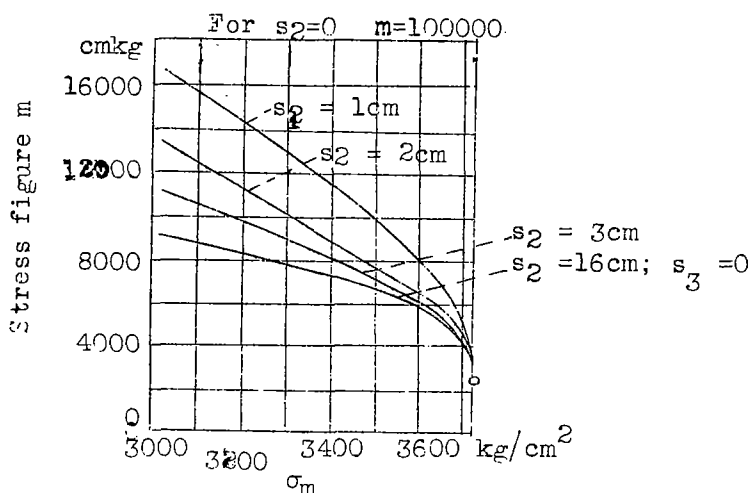
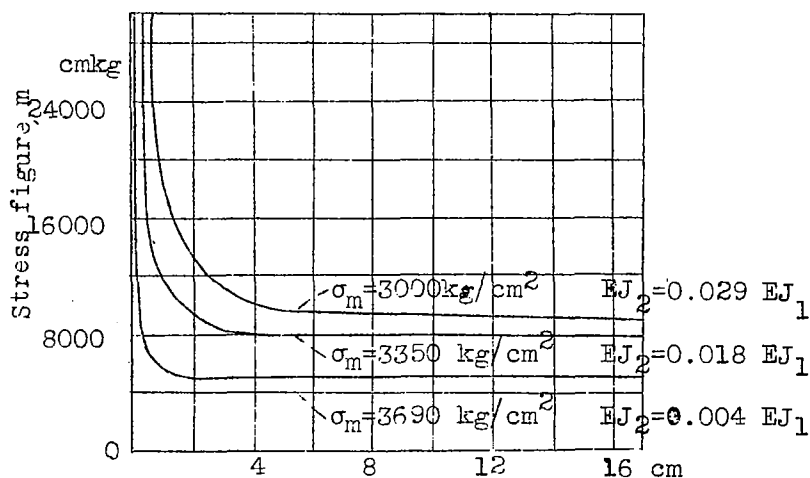


Figure 10.-  $m$  versus  $\sigma_m$  for different lengths  $s_2$  of part stressed beyond elastic range corresponding to figure 9.



Length  $s_2$  part stressed beyond the elastic range.

Figure 11.-  $m$  versus  $s_2$  corresponding to figure 9 for different  $EJ_2$ .

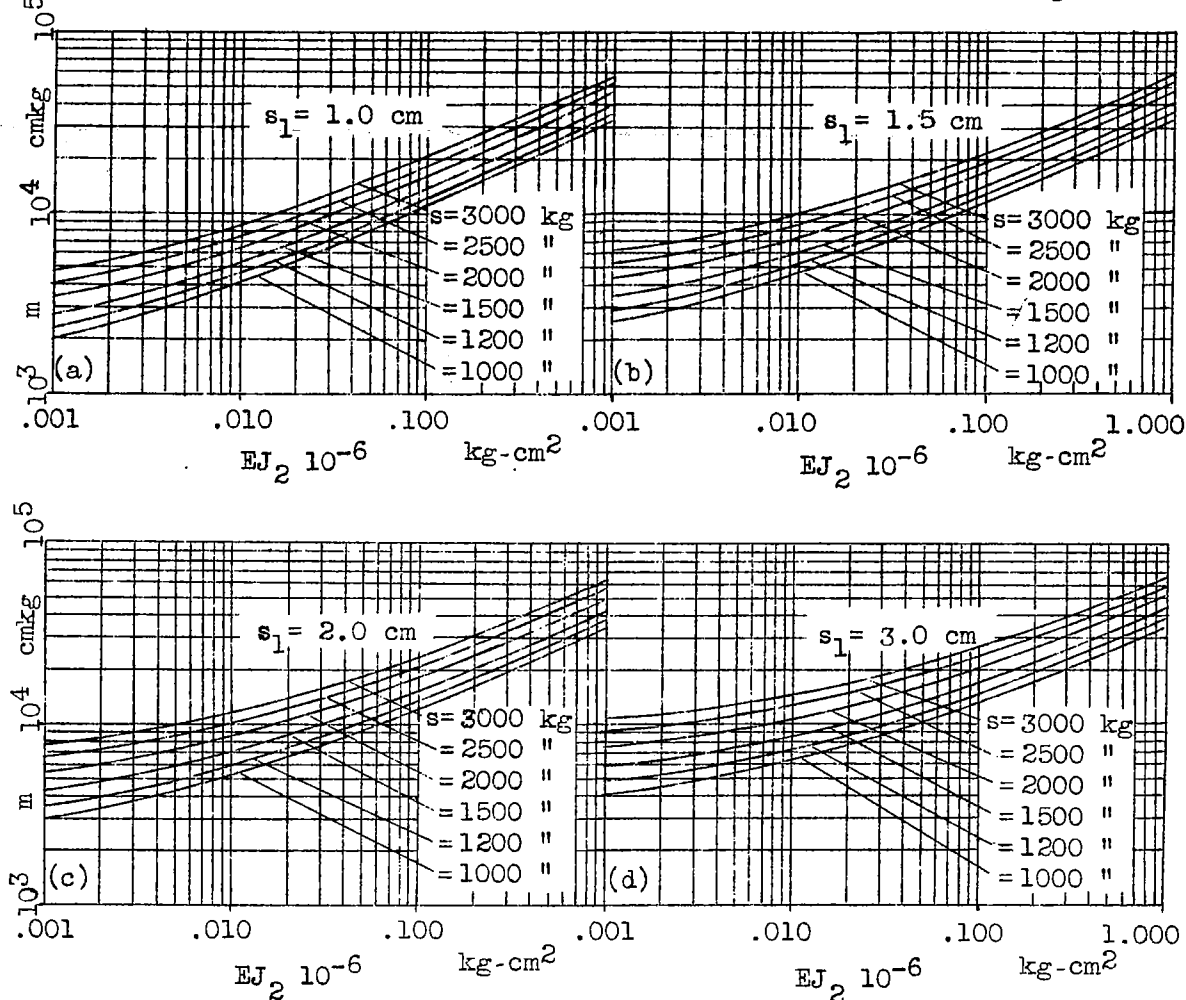


Figure 12(a-d).—  $m$  versus  $EJ_2$  for different axial loads  $S$  with validity of approximation formula  $m = S(s_1 - k_2)$  for four different lengths of elastic margin  $s_1$ .

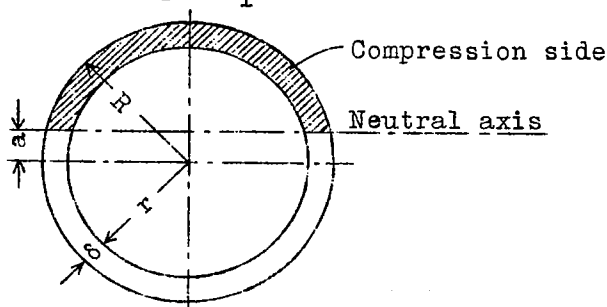


Figure 13.— Location of neutral axis in section for tension beyond elastic range superposed by bending ( $R - a > \delta$ .)



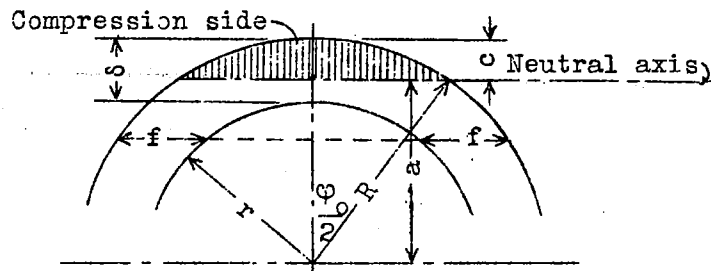


Figure 14.- Location of neutral axis in section for tension beyond elastic range superposed by bending. ( $R-a < \delta$ .)

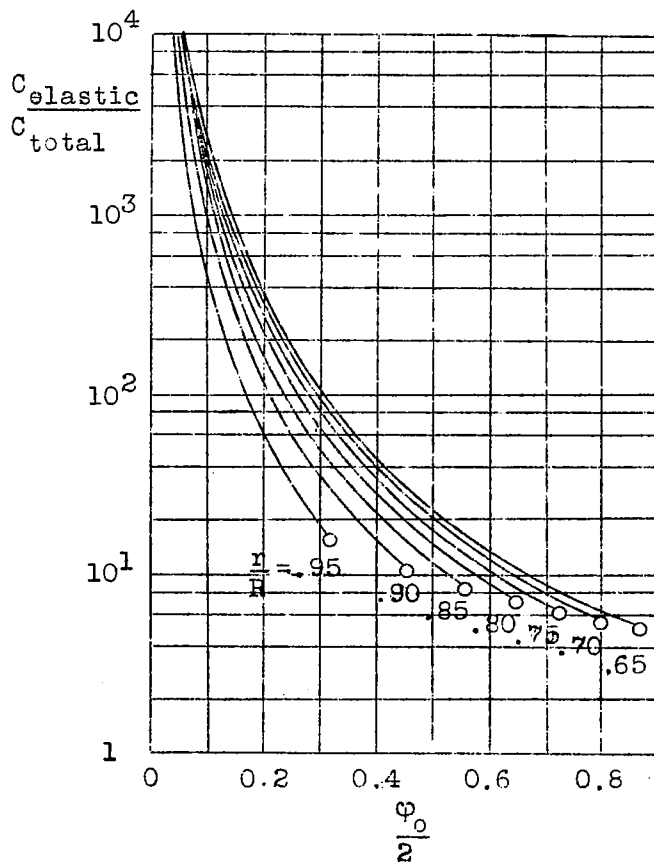


Figure 15.- Curves for defining  $\varphi_0$ .

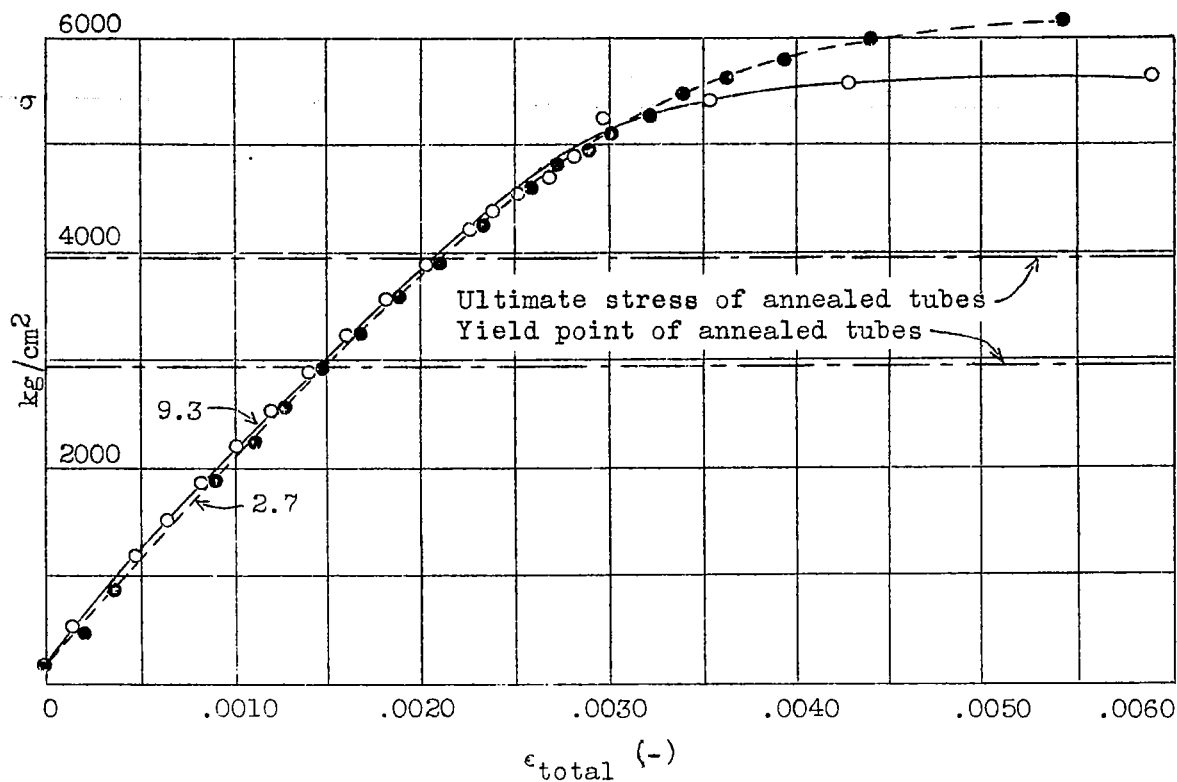


Figure 16.- Tension  $\sigma$  versus total elongation  $\epsilon_{\text{total}}$  for two smooth drawn sample tubes.

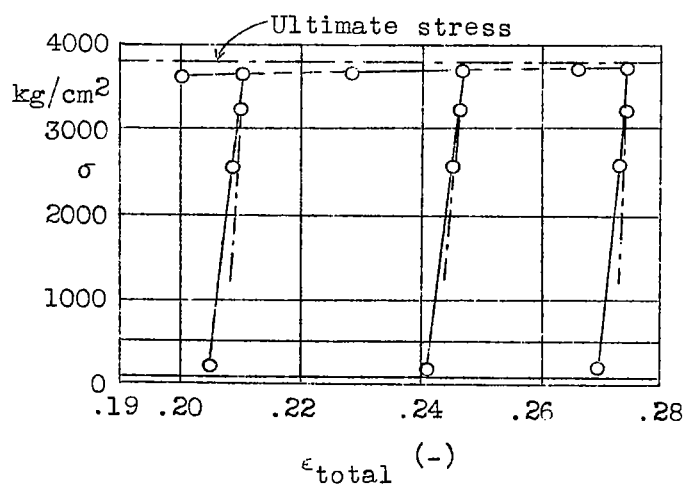


Figure 17.- Return curves of hysteresis loop for annealed tubes near breaking strain.

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